Library,

Henry Phillips, Jr., Edwin J. Houston, William V. McKean, Thomas H. Dudley, John R. Baker.

The Treasurer presented the annual report of the Trustees of the Building Fund.

And the meeting was adjourned.

The limits of stability of nebulous Planets, and the consequences resulting from their mutual relations. By Prof. Daniel Kirkwood.*

(Read before the American Philosophical Society, November 21, 1884.)

To determine the height of the atmosphere is a problem of no common difficulty. This is evident from the fact that estimates derived from the phenomena of twilight, luminous meteors, and the aurora borealis have been widely various. It cannot extend, however, beyond the limit at which its elasticity is counterbalanced by the force of gravity—a limit probably not less than two hundred miles from the earth's surface. Even the volume and weight of this atmospheric envelope are not absolutely constant, as small quantities of gaseous matter are doubtless brought into it from time to time by meteors and meteoric streams. Nor has this accession of matter from without been the only source of variation ; it has been shown by several writers that the extent and density during the cycles of geologic time were in all probability much greater than at present.

But whatever the mass or density of the earth's gaseous envelope, an absolute limit—corresponding to the earth's present time of rotation—may be assigned it. "The atmosphere," says Laplace, "can only extend itself at the equator to the point where the centrifugal force exactly balances the force of gravity ; for it is evident that beyond this limit the fluid would dissipate itself." This limit for the earth is 26,240 miles from the centre ; for Saturn it is within the system of rings ; and for the sun it is at the distance of sixteen millions of miles. These distances, however, were obviously greater before the members of the system had contracted to their

* A preliminary discussion of equation (1) in the following paper was given in the Analyst for January, 1881. Those solutions are here revised, and the results for each planet carefully determined.

present dimensions. It is now proposed to find their original or maximum values.

In astronomy, as in other branches of physical science, many well-known facts remain still unexplained. This is true not only in regard to the fixed stars and the nebulæ, but within the narrower limits of the solar system. Recognizing the impossibility of accounting for present relations without considering the causes which operated in the distant past, astronomers have attempted to trace the process of formation from the primal chaos down to the origin of the youngest planet. In the theory of Laplace, the planets were formed from nebulous rings successively abandoned in the plane of the solar equator. The present writer, while not rejecting the nebular hypothesis itself, has indicated certain objections to the special form in which it was proposed by its celebrated author.* These difficulties, encountered in the theory of formation from *rings*, are avoided by supposing each planet at its origin to have been separated from a very limited arc of the equatorial protuberance. In either case, however, the dimensions of the primitive planet would be necessarily restricted by the law of gravitation.

It is sufficiently obvious that an original planetary mass in a nebular state could not have retained its continuity of form beyond a certain determinable limit; in other words, that it would have been changed into a ring by the attraction of the central body. The main design of the following paper, after finding in several cases the limits of equilibrium, is to trace, if possible, certain unexplained facts to their origin in these primitive relations between the various members of the solar system.

Limits of Planetary Equilibrium.

If two nebulous bodies, M and m, revolve about a common centre of gravity, the disturbing force of M on the superficial stratum of m is the difference between the attraction of the former on the nearest point of the surface of the latter and that on its centre of gravity. The same is true, *mutatis mutandis*, in regard to the disturbing influence of m on M. If, then,

$a =$ the distance between the centres of M and m, and

$x =$ the distance from the centre of the former to the limit of equilibrium of the latter, we shall have

$\frac{M}{a^2} =$ the attraction of M on the centre of gravity of m,

$\frac{M}{x^2} =$ that on the nearest point of the surface, and

$\frac{M}{x^2} - \frac{M}{a^2} =$ the accelerating force of M on the portion of the surface of m between the two centres; but as these forces from M and m are in equilibrium, the neutral point, or the limit of m, may be found from the equation

* Proceedings of the American Philosophical Society, Vol. xviii, p. 324, and Vol. xix, p. 15.

$$\frac{M}{x^2} - \frac{M}{a^2} = \frac{m}{(a-x)^2} - \frac{m}{a^2} \qquad \text{(1)}.$$

Applying this equation to the solar system, x will be the equatorial radius, of the solar nebula, and $a - x$ that of a planet at the epoch of its separation. Putting for simplicity $a = 1$, and reducing,

$$x^4 - 2x^3 + \frac{2M}{M-m} x = \frac{M}{M-m} \qquad \text{(2)}.$$

For Jupiter, $m = 1$ and $M = 1048$, hence

$$x^4 - 2x^3 + 2.0019102x = 1.0009551 \qquad \text{(3)},$$

therefore

$$x = 0.92501,$$
$$1 - x = 0.07499,$$
$$(1 - x) \times 489,000,000 = 35,995,200.$$

Solving equation (2) in like manner for each of the principal planets we obtain the distance from the centre of each to its limit as given in the following table:

Planet.	Dist. to Limit.	
Mercury	152,000	miles.
Venus	700,208	"
Earth	1,082,147	"
Mars	764,650	"
Jupiter	35,995,200	"
Saturn	44,887,000	"
Uranus	48,915,000	"
Neptune	81,000,000	"

In these estimates we neglect the eccentricity of the orbits as well as the centrifugal force due to each planet's rotation. The masses and distances adopted are those given in Newcomb's Popular Astronomy, with the exception that for Mercury we have employed a mean between Von Asten's evaluation of the mass $\left(\frac{1}{7,636,440}\right)$ and the final value given by Leverrier $\left(\frac{1}{5,310,000}\right)$. The mean is $\frac{1}{6,263,800}$. For the earth we have taken the sum of the masses of the earth and the moon.

Applying equation (2) to some of the secondary systems we find the following limits of stability:

For the Moon	39,850	miles.
Phobos	6.5	"
First satellite of Jupiter	5,250	"
Mimas	1,500(?)	"

Practical Applications.

The results obtained may now be employed in the approximate solution of several interesting problems. The limits of stability will be regarded as the primitive radii of the planets and satellites, as any exterior matter would have been detached by the influence of the central body. To the

primitive relations above developed may we not hope to trace some of the unexplained facts of the solar system? As has been remarked by an eminent writer,* "the plan of the coming universe must have resided in the initial chaos, as certainly as the eagle is in the egg, or the leviathan in its primitive germ."

I. To find the relative mean densities of the earth and moon at the epoch of their separation.

With the notation used in equation (1) the ratio sought will evidently be

$$\frac{\rho M}{x^3} : \frac{m}{(a-x)^3},$$

where $\rho =$ the ratio of the equatorial to the polar radius of the terrestrial spheroid. The value of this ratio is not known. An approximate value may be found, however, by a tentative process.

We have $a = 240,300$ miles, $x = 200,450$, $a - x = 39,850$, $M = 81$, and $m = 1$. Hence the ratio is $0.636\rho : 1$.

But during the cooling period the ratio of the densities would probably be nearly constant; or, if the moon contracted more rapidly its solidification would occur earlier and the increase of its density practically cease. The present ratio of the mean densities is 5.67 : 3.57, and assuming this to have been constant we obtain

$$0.636\rho : 1 :: 5.67 : 3.57,$$

or,

$$\rho = 2.498;$$

that is, the ratio of the earth's equatorial to its polar radius at the epoch of the moon's separation was nearly 5 : 2, and this may be regarded with some probability as nearly the ellipticity in other cases at the respective epochs of separation.

II. To find the relative mean densities of Jupiter and his first satellite at the epoch of the latter's origin.

Here $a = 260,000$ miles, $x = 254,750$, $a - x = 5250$; and therefore the ratio is

$$\frac{59240\rho}{(254750)^3} : \frac{1}{(5250)^3} = 0.52\rho : 1,$$

and assuming the constancy of the ratio,

$$0.52\,\rho : 1 :: 121 : 100; \text{ or, } \rho = 2.33.$$

This value of ρ is nearly equal to that found for the earth; the difference being no greater than might result from the probable error in the elements used.

The present density of Phobos is unknown; but with $\rho = 2.5$, the value found for the earth, the ratio of the original densities of Mars and Phobos was 1.27 : 1. These results seem to indicate that the ratio of the equatorial to the polar radius of the central mass, at the epoch of a planet's or satel-

* Prof. Pierce.

lite's origin, was about 2.5.* With this value of ρ, and the value of x already obtained for each planet, the ratio of the mean density of the solar mass to that of the planets at the respective epochs of their separation would have been as follows:

For Neptune	1.31 : 1
" Uranus	1.31 : 1
" Saturn	1.39 : 1
" Jupiter	1.39 : 1
" Mars	1.25 : 1
" Earth	1.29 : 1
" Venus	1.27 : 1
" Mercury	1.22 : 1

From these numbers we infer that central condensation had commenced in the solar nebula before the origin of Neptune, and that the ratio of the mean density to the density of the equatorial parts near the surface was approximately the same at the successive epochs of planetary formation.

Were the Planets formed from nebulous Rings?

If the original solar mass, like most nebulæ, was irregular in form, the first matter detached would not probably be a ring, but a nebulous planet. As condensation advanced, the centrifugal force would increase until approximately equal to the central attraction. The disturbing influence of the planet already formed would produce, when in perihelion, an increasing tidal-wave, resulting in the separation of a second planet. The origin of other planets is accounted for in like manner. If, in the ancient history of the system, nebulous matter, left at first exterior to the orbit of a new planet, should subsequently fall upon the central body, the effect would be not only a shortening of the period, but probably also a lessening of the orbit's eccentricity.

III. *The Peculiar Relations of the Martian System.*—Professor Pickering estimates the diameter of Phobos at seven miles.† Adopting this value, and supposing the ratio between the densities of Phobos and Mars equal to that between the moon and the earth, we shall find the limit of the satellite's equilibrium to be 6.5 miles from its centre, or three miles from its surface. Were the density reduced to that of Saturn, the limit would be almost exactly at the surface; or, with a density equal to that of Mars when the radius of the latter was that of the satellite's orbit, the limit would be at a considerable distance within the surface. Since, therefore, the satellite could never have existed at its

* It was shown by Laplace that a rotating homogeneous fluid cannot retain its spheroidal form when ρ is greater than 2.7197. Mec. Cel. III, *iii*, §20 [1605'], Bowditch's Trans. The ratio would be less in the case of central condensation.

† Annals of the Observatory of Harvard College, Vol. xi. Professor Seth C. Chandler makes the diameter still less. See Sci. Obs. for Sept., 1877.

present distance in a nebular state, it must follow, if any form of the nebular hypothesis is to be accepted, that its original distance was greater than the present. Can we assign a probable cause for this ancient disturbance?

Of the eight major planets, Mars has the most eccentric orbit, except that of Mercury; its perihelion distance being 13,000,000 miles less than its mean distance. This difference, in fact, amounts to 20,000,000 miles when the orbit of Mars has its greatest eccentricity. If, therefore, the radius of the sun, or of the solar atmosphere, was somewhat greater than the least distance of Mars at the commencement of the latter's separate existence, the planet in perihelion would pass through the outermost equatorial zone of the solar nebula. This resisting medium would not only accelerate the motion of Mars, but also, in a much greater degree, that of his extremely small satellites. The solar volume, meanwhile contracting more rapidly than the orbit of Mars, would finally leave the latter moving in an eccentric path, without sensible resistance.*

IV. *The Saturnian System.*—For Mimas, the first satellite of Saturn, the most probable values of the mass and density give the distance of the limit from the satellite's surface less than the radius of Mimas. The rings of Saturn, in all probability, could not exist as three satellites, the limits of equilibrium being interior to the surface. This is true at least in the case of the innermost ring. Analysis seems to indicate that PLANETS AND COMETS HAVE NOT BEEN FORMED FROM RINGS, BUT RINGS FROM PLANETS AND COMETS. If, without any loss of mass, the density of a planet were diminished until the radius should exceed the limit of equilibrium, what change would take place in the planetary form? Evidently a portion of the matter nearest the central body would be separated from the rest, and, as the orbital velocity would be less than that corresponding to its distance, it would move in a new ellipse, the aphelion of which would be the point of separation.

V. *Comets.*—The effect of the sun's attraction in the dismemberment of comets is well known to astronomers. The nuclei of the large comets of 1680, 1843, 1880 and 1882 must have had great force of cohesion between their parts, in order to withstand the tendency to disintegration at the times of perihelion passage. Had the nuclei been either liquid or

* This view was first presented in the Observatory for January, 1878. Different explanations of the short period of Phobos have been proposed by astronomers, but none, perhaps, entirely free from difficulties. One distinguished writer has suggested that $7^h\ 39^m$, the period of Phobos, was the rotation period of Mars at the epoch of the satellite's origin, and that the lengthening of the period to $24^h\ 37^m$ has been due to retardation by solar tides. But it is well known that the time of rotation of a planet in the process of condensation varies as the square of its radius. The resulting period of Mars, therefore, on reaching its present dimensions, would have been but a small fraction over one hour. This period, it is true, would have been somewhat modified by the counteracting influence of the solar tides; but the hypothesis referred to seems wholly inadequate to meet the objection derived from equation (2).

gaseous, or even clusters of solid meteorites, the difference between the sun's attraction on the central and the superficial parts would have pulled the comets asunder, spreading out the fragments into somewhat different orbits, like the meteoric streams of August and November.

This view of the gradual dispersion of comets in perihelion is in striking harmony with the facts of observation. The comets of short period have not only been divested of tails, which in all probability they originally possessed, but they seem to be losing more and more of the cloud-like matter which surrounds their nuclei. Halley's comet has lost much of its ancient splendor, and, had its period been no greater than that of Encke's or Biela's, it might long since have been reduced to a telescopic magnitude. The separation of Biela's comet, in 1845, was not the beginning of that body's dismemberment. We have evidence that this process had commenced before 1798, as in that year a meteoric shower, produced by its *débris*, was observed in Europe. A shower derived from the same group was again seen in 1838.* Before 1845, however, the separated fragments were too small to be individually recognized. How far the sun's action alone can explain the facts, it may be impossible to determine.

VI. *The Zodiacal Light.*—Original small planets near the sun, in a nebular or gaseous condition, would probably be transformed either into rings or meteoric clusters, the scattered particles of which, reflecting the sun's rays, would present an appearance like that of the zodiacal light.

VII. *Origin of the Asteroids.* — In the primitive condition of a planet, immediately after its separation from the central mass, not only would the latter cause a considerable elongation of the former in the direction of the line joining their centres, but the planets also—especially the larger—would produce great tidal elevations on the sun's surface. Now, a comparison of the elements of Hilda and Ismene, the 153d and 190th asteroids, shows them to be an isolated pair whose periods are very nearly equal, each exceeding the longest in the interior cluster by more than fifteen months. Jupiter's limit of equilibrium, when in the nebular form, was immediately beyond the orbits of these minor planets. If the sun once extended to the aphelion distance of Hilda (4.632), the central attraction of his mass on a particle of the equatorial surface was but five times that of Jupiter at the point to which he was vertical.† The centrifugal force due to the sun's rotation would be greatest at the crest of this tidal wave, produced by Jupiter, so that parts might become separated from the solar mass, and transformed into asteroids. It is to be further remarked that two periods of Jupiter are approximately equal to three of Hilda and Ismene, that is, to three rotation periods of the sun at the epoch of their separation. The disturbing effect of the "giant planet" on the tides of the central body would therefore be increased at each perihelion passage.‡

* Humboldt's Cosmos, Bohn's ed., Vol. iv, p. 582.

† Jupiter's perihelion distance is 4.95.

‡ The longitude of Ismene's perihelion differs from that of Hilda's by 180°.

The process would be similar when one period of Jupiter was equal to two rotation periods of the central nebula.

VIII. *The Rotations of the Planets.*—It is well known that the analogy between the periods of rotation of the primary planets, as published by the present writer several years since, assigned a much longer period of rotation to Uranus than to Jupiter or Saturn. But as that of Uranus had not been measured, and the observations of the polar compression were by no means accordant, the fact was not then thought incompatible with the proposed law of rotation. Recent measurements, however, leave no room to doubt that the ellipticity is even greater than that of Jupiter, and consequently that the planet moves rapidly on its axis. The law connecting the rotation periods must accordingly require an important modification.

In a planet having a constant mass, with a variable volume, the time of rotation varies as the square of the radius. It is easy to show, however, that this law could not have obtained from the origin of the solar system. For instance, in tracing backward the history of the earth, we find that when the radius was 8000 miles, its rotation period, according to this law, was 96 hours; when the former was 12,000 miles, the latter was 216 hours; and, finally, if the earth ever extended to the moon's orbit, the time of rotation, by the same law, instead of having been equal to the moon's orbital period, was nearly ten years. So likewise when Mars filled the orbit of Phobos, his rotation period was 7 days and 16 hours, or 24 times the orbital period of the satellite. We conclude, therefore, that during the earlier stages of its condensation all parts of the mass did not rotate in the same time. It is easy to see, in fact, that tidal retardation must have been much more effective at the surface than in the interior of a large planet in the gaseous state.

In so far as we know, the rotation period of the smaller planets, Mercury, Venus, the Earth and Mars, are nearly two and a half times those of the larger and more remote. What cause can be assigned for this remarkable difference? In other words, why did the process of condensation continue longer in the large and less dense planets exterior to the asteroids than in the small bodies nearer the sun? It may be answered in a general way that in small and dense planets solidification would occur at a comparatively early epoch in their history, and hence the acceleration of their rotary velocity would be, in a large measure, arrested. It seems probable, therefore, that, while the same law of rotation may obtain between the members of each separate group, it cannot apply where one of the planets is in the inner and the other in the outer cluster.

As regards their axial movements, the solar system appears to contain at least three distinct classes of planetary bodies; the obvious characteristics of each being traceable to their relative primitive densities. These are as follows, the primitive density of Neptune being taken as unity:

I The large planets :

	Primitive Density.
Neptune	1.0000
Uranus	3.8950
Saturn	32.7073
Jupiter	210.7440

II. The planets interior to the zone of asteroids :

Mars	7,446.4
Earth	24,880.5
Venus	70,129.2
Mercury	468,616.0

III. The secondary planets, of which our moon and Jupiter's first satellite may be taken as types :

Jupiter's first satellite	2,600,000
The Moon	4,820,000

There is, we may remark, an antecedent probability that the law truly formulated will assign to Saturn a period of rotation somewhat less than the period observed; as it is sufficiently obvious that if the ring had remained an integral part of the planet, the resulting time of rotation would have been, in fact, sensibly shorter than the present. It is also to be remembered that the late observations of Denning and Schiaparelli make Mercury's time of rotation nearly 25 hours. In the case of the satellites, the equality between the periods of rotation and revolution was established at an early epoch in their history. No further decrease in the time of rotation was therefore possible.

A comparison of the quantities used in equation (1) suggests that a planet's time of rotation is a function of its mass, distance, and primitive density. The form of this function—found by a tentative process—may be expressed as follows :

The square of the number of a planet's days in its year is to that of any other of the same group, as the primitive density of the latter is to that of the former ; that is,

$$n^2 : n' :: \triangle' : \triangle \,; \text{ or, } n' = n \left(\frac{\triangle}{\triangle'}\right)^{\frac{1}{2}} \quad \ldots\ldots (4),$$

where

$$\triangle = \frac{m}{R^3} = \text{the primitive density, and}$$

$$n = \frac{T}{t} = \frac{\text{orbital period}}{\text{rotation period}} = \text{the number of a planet's days in its year.}$$

Equation (4) may be reduced to

$$t^2 : t'^2 :: m \left(\frac{d}{R}\right)^3 : m' \left(\frac{d'}{R'}\right)^3 \quad \ldots\ldots (5),$$

where d, d' and R, R$'$ are the respective distances and primitive radii.

The Outer or less dense Group.

In the following table the rotation periods of Saturn, Uranus and Neptune are derived from that of Jupiter by formula (4).

Planet.	m	R	$\triangle$	T	t		n
Neptune.	16.86	81,000,000 m	1.0000	60,126.71 d	9 h	35 m	150,751.80
Uranus ..	14.46	48,915,000	3.8950	30,686.82	9	33	77,186.00
Saturn...	93.84	44,887,000	32.7073	10,759.219	9	43	26,630.00
Jupiter ..	311.80	35,995,200	210.7440	4,332.584	9	55	10,492.64

It will be noticed that the theoretical period of Saturn is 31 minutes less than Hall's evaluation.

The Inner Group.

Planet.	m	R	$\triangle$	T	t			n
					h	m	s	
Mars	0.1056	764,650 m	7,446.40	686.98 d	24	37	23	669.57
Earth. ...	1.0000	1,082,147	24,880.5	365.26	23	53	48	366.84
Venus...	0.7690	700,208	70,129.2	224.70	24	42	54	218.18
Mercury .	0.0522	152,000	468,616.0	87.97	25	0	46	84.405

Here the rotation periods of the earth, Venus and Mercury are derived from that of Mars by formula (4). The first is two minutes less than the true period; the time of Venus's rotation is doubtful; and the theoretical determination of Mercury's period agrees with the estimate of Mr. Denning.

ERRATA IN PROF. KIRKWOOD'S ARTICLE.

Page 106, erase the comma at the end of l. 2 from top.

" " l. 10 from top, for 489,000,000, read 480,000,000.

Page 111, l. 16 from bottom, for period read periods.

Page 112, l. 7 from bottom, for n' read n^2.